An exact analysis of a switched-capacitor circuit (°)

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Abstract. A model for a Switched-Capacitor (SC) is introduced, which permits a simple and accurate analytical solution of SC networks. The theoretical and experimental results obtained for a simple low-pass filter check the validity of the model and show the approximations needed for a SC circuit to equate the RC corresponding version. We have calculated and measured the response to a sinusoidal signal and to noise, showing the buildup of the output noise due to aliasing effects.

1. INTRODUCTION

The use of a Switched-Capacitor (SC) to simulate a resistance in filter design is well established, due to advantages obtainable by using MOS integrated technology. Different arrangements for a capacitor and associated switches have been suggested [1], [2]; more recently, with the aim to realise a bilinear transformation between the s-plane and the z-plane, a particular circuit has been proposed [3] (Fig. 1a).

In this work, we have analyzed in the Laplace or Fourier domain a simple low-pass filter with the SC element, analogous to a RC circuit in the continuous case. We have obtained the exact output response for a general input signal with an analytical method that, for the SC circuit considered [3] is more direct and simple than the one of Lin and Kuo [4]. The results are useful to evaluate the noise properties of a SC active filter due to internal noise sources as those associated with the operational amplifiers. We have therefore obtained the fundamental frequency response and the output frequency spectrum for a sinusoidal input signal; consequently the output noise power spectrum corresponding to input noise not bandlimited is derived. An experimental analysis has checked the validity of the theoretical results.

2. CIRCUIT ANALYSIS

In the SC circuit of Fig. 1a, we assume that the switches commute every T seconds, changing instantaneously from AA' to BB' (or viceversa).

Fig. 1 a) The Switched Capacitor,
b) The equivalent circuit.

The capacitor C₁, therefore, is always connected to the external circuit, except (momentarily) at the switching times t=T, when it is reversed. The action of the four switches can be modeled with a f-current generator placed across the capacitance C₁; at times t=T, pulses having charges 2C₁Vₘ₃ are injected in parallel to the capacitor, to simulate its periodical reversing. Therefore, the circuit shown in Fig. 1b is the equivalent one of the SC of Fig. 1a and it is suitable for the analysis of general circuits including SC sections.

In this work we use this equivalent model for solving the simple low-pass SC filter of Fig. 2, and we compare its behavior with the corresponding analog RC circuit.

In the analysis of the time-dependent circuits implicit relations between the Laplace transforms of the input and output signals can be derived. The mathematical analysis reported in Appendix,

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Fig. 2 The passive SC low-pass filter.

and useful for ideal sampling, leads to an explicit expression for the output voltage as a function of the input voltage in the Laplace or Fourier domain. It represents a straightforward tool to solve networks embedding SC elements of the type shown in Fig. 1a.

In our case for the low-pass SC filter we obtain (see Appendix eq. 8A):

\[ V_i(s) = \frac{a}{1-a} V_o(s) \left( \frac{1 - e^{-sT}}{sT} \right) \]

where \( V_i(s) \) and \( V_o(s) \) are the Laplace transforms of the \( v_i(t) \) and \( v_o(t) \) signals, and \( V_i(s) \) is the Laplace transform of the selected signal \( v_i(t) = \sum_{n=0}^{\infty} v_i(nT) \delta(t-nT) \), that is:

\[ V_i(s) = L[v_i(t)] = \sum_{n=0}^{\infty} V_i(s - \frac{j2n\pi}{T}) \]

\( a \) is the ratio of the capacitance \( C_1/C \).

3. FREQUENCY RESPONSE

For the particular case of a sinusoidal input using the Fourier transform, that is with \( s = ju \), we have:

\[ v_i(t) = \cos(\omega t) \]

\[ V_i(ju) = \frac{a}{1-a} \left( \delta(\omega - u) - \delta(\omega + u) \right) \]

and from (1):

\[ V_o(ju) = \frac{a}{1-a} \sum_{n=0}^{\infty} \left[ \delta(\omega - u - \frac{2\pi n}{T}) - \delta(\omega + u + \frac{2\pi n}{T}) \right] \]

The output \( V_o(ju) \) consists of sinusoidal terms at frequencies \( \omega, \omega \pm 2k\pi/T \), \( k = 0, 1, 2, \ldots \).

If the component at the fundamental frequency \( u_0 \) is separated we can write:

\[ V_o(ju) = \tau H_1(ju) \left[ \delta(\omega - u) - \delta(\omega + u) \right] + \frac{\alpha}{1-a} \sum_{n=0}^{\infty} \left[ \delta(\omega - u - \frac{2\pi n}{T}) - \delta(\omega + u + \frac{2\pi n}{T}) \right] \]

where:

\[ H_1(ju) = \frac{\alpha}{1-a} \left[ 1 + \frac{2}{T} \frac{1}{1-a} \frac{1 - e^{-juT}}{juT} \right] \]

\[ H_1(ju) = \frac{2a}{(1-a)} \frac{1 - e^{-juT}}{juT} \left[ 1 + \frac{1 - e^{-juT}}{1+a} \right] \]

For analogy with continuous linear circuits we can define the term \( H_1(ju) \) the frequency response. This is, by inspection of (4), the output signal at the same frequency of the input.

\( H_1(ju) \) evaluated for \( \omega = u, \omega - 2n\pi/T, n\pi \), gives the output component at that frequency for input at \( u \) (that is for \( v_i(t) = \cos(\omega t) \)).

In (5) the first term accounts for the capacitive feedthrough, the second one describes the low-pass behaviour of the circuit. In fact, if we consider (5) for \( \omega T < 1 \), we have:

\[ H_1(ju) = \frac{\alpha}{1-a} \left[ 1 + \frac{1}{\mu \frac{1}{1+a} \frac{1 - e^{-juT}}{juT}} \right] \]

The second term of (7) equals the response of a RC low-pass circuit. If \( \omega T \approx 1 \) the feedthrough is negligible, in the interesting frequency range, and the (7) equals the expression of an RC circuit with \( RC = T/2a = T/C_1/C \), stating the well known correspondence R-T/2C between continuous and sampled cases.

By utilizing (5) we have calculated the frequency response for \( \alpha = 0.5, 0.05, 0.005 \) and for sampling frequency 1/T = 1kHz; 10kHz; 100kHz respectively. In this way we maintain the same equivalent resistances R-T/2C. The theoretical results are drawn in Fig. 3a and 3b as amplitude and phase. In the same figures we report the response of the analog RC circuit, with \( RC = 10^{-3} \).

It is evident that the SC circuit behaves like the analog RC circuit in a definite frequency range. The best performance is achieved for the largest sampling frequency, that is for \( C_1/C = 0.005 \) and 1/T = 100kHz; this implies that the integrating frequency interval must lie in the range where \( \omega T < 1 \); beyond this limit also for the feedthrough effect, the integrative behaviour of the SC circuit is degraded.

The experimental set-up consists of a switched capacitor \( C_1 \) of 1nF (the values of \( C \) are determined by the selected \( \alpha,C_1/C \) ratios); the switches are analog Bipolar transistors LF 113330 integrated circuit, suitable to operate in the chosen frequency range. The output voltage is measured with a spectrum analyzer. The experimental results reported in Fig. 3a show good agreement with the
calculated response.

![Graph of calculated response](image)

4. NOISE

The voltage output power spectra $S_o(\omega)$ due to a voltage noise generator at the input $S_o(\omega)$ can be obtained summing up at the output frequency $\omega$ different contributions arising from the input noise at frequencies $\omega \pm 2n\pi/\nu$ (n=0,1,2,...). In fact, the output noise at $\omega$ comes out by a contribution of the input noise at the same frequency, and the noise due to aliasing, that is the input noise at $\omega \pm 2n\pi/\nu$ converted at $\omega$. It is important to observe that in SC filters the internal noise sources, mainly those due to operational amplifiers, cannot be prefiltered in the Nyquist band as it is done for external signals. Therefore, it is necessary to consider the actual noise spectrum and the aliasing effects.

Being the input noise at different frequencies uncorrelated, we have:

$$S_o(\omega) = S_0(\omega)|H_o(j\omega)|^2 \sum_{n=-\infty}^{\infty} S_n(\omega + 2n\pi/T)$$

The output noise $S_o(\omega)$ depends critically on the input spectrum inside and outside the Nyquist band.

If $\omega << 1$, $H_o(j\omega)$ and $H_1(j\omega)$ practically coincide within the Nyquist band, and (8) becomes:

$$S_o(\omega) = |H_1(j\omega)|^2 \sum_{n=-\infty}^{\infty} S_n(\omega + 2n\pi/T)$$

The baseband noise at $\omega$ sums up contributions from different frequencies, aliased in the baseband. As an interesting example, if we assume the input spectrum white and equal to $S_0$ till to $\omega_H$ and zero for higher frequencies, the noise spectrum in the Nyquist band, under the condition $\omega << 1$, is approximately:

$$S_o(\omega) \approx \sum_{n=-\infty}^{\infty} S_n(\omega + 2n\pi/T)$$

That corresponds to the usual case ($\omega >> \omega_N$), to a substantial excess noise factor (given by the ratio $\omega_H/\omega_N$) with respect to the output noise of the equivalent RC circuit. The excess noise outside the Nyquist band can be determined by (8). It is, in particular, zero at $\omega = 2n\pi/\nu$ and it is always lower than the amount given by (10).

Fig. 3 - Amplitude (a) and phase shift (b) of $H_1(j\omega)$ for $\alpha$ = 0.5, 0.05, 0.005 and sampling frequencies of 1kHz, 10kHz, 100kHz respectively. The continuous line refers to theoretical result; the points refer to experimental results; the dotted line refers to the analog corresponding RC circuit with RC = $10^{-3}$.

Fig. 4 - Theoretical and experimental values of $|H_1(j\omega)|$ v.s. the normalized frequency $\nu$ for $\alpha$ = 0.5, 0.05, 0.005.

With the same experimental set-up we have measured the harmonic response to check eq. (6). For a sinusoidal input signal at $\omega$ we have measured the output voltage at the different harmonic frequencies $\omega_0 + 2n\pi/\nu$ (n=0). In Fig. 4 we have reported $|H_1(j\omega)|$ given by (6), for different $\alpha$ ($\alpha$ = 0.5, 0.05, 0.005) as a function of $\omega/T$. In the same figure the experimental points are shown.

Fig. 4 - (a) Theoretical and experimental values of $|H_1(j\omega)|$ v.s. the normalized frequency $\nu$. (b) Theoretical and experimental values of phase shift for $\alpha$ = 0.5, 0.05, 0.005.

In Fig. 5 we report the theoretical as given by (8) and experimental behaviour of the output spectrum $S_o(\omega)$ due to input white noise band limited to 22kHz ($\omega_H = 138.6$kHz) and sampling rate of 1kHz ($\omega_N = 3.14$kHz). In the same figure we report, with dotted lines $|H_1(j\omega)|^2$. It can be verified that the region $\omega << \omega_N$ the excess noise for $\alpha = 0.005$ is 10dB as obtainable from (10), and
for \( a = 0.5 \) the excess noise is lower and equal to 13dB, according to (5).

\[
\text{Fig. 5 - The output noise spectra for different } a \text{ when the input noise } S_{1}(\omega) \text{ is white and band-limited to } 22kHz; \text{ the sampling frequency is } 1kHz.
\]

In the experimental set-up the noise generator was Rohde & Schwarz Mod. BN4150, filtered to 22kHz with Krohn-Hite filter mod. 3322; the output noise was measured by HP 8553B spectrum analyzer.

5. CONCLUSION

The analysis of networks with switched capacitor circuits like that of Fig. 1A, can be simply and exactly done through the equivalent circuit characterized by a voltage controlled time-varying current generator. Moreover, such time-varying networks can be analytically resolved in the Fourier domain to get output signal, whichever the input signal is.

The theoretical and experimental results obtained check the validity of the assumed model and show the necessary approximations for a SC circuit to simulate a low-pass RC cell.

As far as the noise is concerned, it is shown the importance of the aliasing effect in building up the output noise; this effect can be very important in connection with SC active networks, because the operational amplifiers noise bandwidth normally exceeds the Nyquist frequency of the switched network.

APPENDIX

The circuit of Fig. 1A is the equivalent one of the SC low-pass filter where \( i(t) \) is given by:

\[
(1.A) \quad i(t) = 2C \sum_{n=-\infty}^{\infty} \left[ v_{i}(nT) - v_{o}(nT) \right] e^{j2\pi n \frac{t}{T}} \tag{1.A}
\]

\( v_{i}(t) \) and \( v_{o}(t) \) are assumed to be continuous at times \( nT \). Taking the L-transform of (1.A), we get:

\[
(2.A) \quad L(i(t)) = \frac{2C}{T} \sum_{n=-\infty}^{\infty} \frac{1}{s} \left[ v_{i}(nT) - v_{o}(nT) \right] e^{j2\pi n \frac{t}{T}}
\]

\[
= \sum_{n=-\infty}^{\infty} V_{i}(s) - V_{o}(s)
\]

\[
= \frac{2C}{T} \left[ V_{i}(s) - V_{o}(s) \right]
\]

\[
\text{Fig. 1A - The equivalent circuit for the SC low-pass filter of Fig. 2.}
\]

In (2.A) we have introduced the symbol \( \delta \) to denote signals sampled by ideal sampling, that is:

\[
(3.A) \quad A(s) = \sum_{n=-\infty}^{\infty} A(s-j\frac{2\pi n}{T})
\]

\[
= T \cdot (\sum_{n=-\infty}^{\infty} a(nT) \delta(t-nT))
\]

For the sampler operator is valid the following rule [5]:

\[
(4.A) \quad (A \cdot B) = A \cdot B
\]

being \( A \) and \( B \) the L-transforms of the two functions \( a(t) \) and \( b(t) \) assumed continuous for \( t-nT \).

By inspection of the circuit given in Fig. 1A, we get:

\[
(5.A) \quad V_{i}(s) = \frac{C_{i}}{sC_{i} + 1} V_{i}(s) + \frac{1}{s(C_{i}C_{o})} I(s)
\]

\[
= \frac{a}{1-a} \left[ V_{i}(s) - \frac{2}{sT} \left[ V_{i}(s) - V_{o}(s) \right] \right]
\]

with \( a = C_{i}/C_{o} \).

Applying the sampler operator to (5.A) and taking into account (4.A) we obtain:

\[
(6.A) \quad V_{o}(s) = \frac{1}{1-2\frac{1}{sT}} V_{i}(s)
\]

For the Poisson Sum formula we can write:
\[
\begin{align*}
(7.4) \quad \frac{1}{sT} = \sum_{n=0}^{\infty} \frac{1}{nT} e^{(t-nT)} \\
= \sum_{n=0}^{\infty} e^{-snT} = \frac{e^{-sT}}{1 - e^{-sT}}
\end{align*}
\]

being \(1(t)\) the step function defined as

\[
1(t) = \begin{cases} 
1 & t > 0 \\
0 & t \leq 0
\end{cases}
\]

Substituting (7.4) into (6.4) and (6.4) into (5.4), we get:

\[
(8.4) \quad \frac{V_0 - \frac{a}{1-a}}{V_1} \frac{2}{sT} \frac{1}{1-a} \frac{1 - e^{-sT}}{1 - \frac{a}{1-a} e^{-sT}}
\]

REFERENCES


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